Available online at www.sciencedirect.com

ScienceDirect

Discrete Applied Mathematics 155 (2007) 1302–1313

DISCRETE APPLIED MATHEMATICS

www.elsevier.com/locate/dam

Extremal problems on consecutive $L(2, 1)$-labelling☆

Changhong Lu[a, b, *], Lei Chen[a], Mingqing Zhai[c]

[a] *Department of Mathematics, East China Normal University, Shanghai 200062, PR China*
[b] *Institute of Theoretical Computing, East China Normal University, Shanghai 200062, PR China*
[c] *Department of Mathematics and Computer Science, Chuzhou University, Chuzhou, Anhui 239012, PR China*

Received 10 January 2006; received in revised form 23 November 2006; accepted 12 December 2006
Available online 20 December 2006

Abstract

For a given graph G of order n, a k-$L(2, 1)$-labelling is defined as a function $f: V(G) \to \{0, 1, 2, \ldots, k\}$ such that $|f(u) - f(v)| \geqslant 2$ when $d_G(u, v) = 1$ and $|f(u) - f(v)| \geqslant 1$ when $d_G(u, v) = 2$. The $L(2, 1)$-labelling number of G, denoted by $\lambda(G)$, is the smallest number k such that G has a k-$L(2, 1)$-labelling. The consecutive $L(2, 1)$-labelling is a variation of $L(2, 1)$-labelling under the condition that the labelling f is an onto function. The consecutive $L(2, 1)$-labelling number of G is denoted by $\bar{\lambda}(G)$. Obviously, $\lambda(G) \leqslant \bar{\lambda}(G) \leqslant |V(G)| - 1$ if G admits a consecutive $L(2, 1)$-labelling. In this paper, we investigate the graphs with $\bar{\lambda}(G) = |V(G)| - 1$ and the graphs with $\bar{\lambda}(G) = \lambda(G)$, in terms of their sizes, diameters and the number of components.

Keywords: Channel assignment problems; Distance-two labelling; Hamiltonian graphs; $L(2, 1)$-labelling; No-hole coloring

1. Introduction

The problem of vertex labelling with a condition at distance two, proposed by Griggs and Roberts [9], arose from a variation of channel assignment problem introduced by Hale [10]. Suppose a number of transmitters are given. We must assign a channel to each of the given transmitters such that the interference is avoided. In order to reduce the interference, any two "close" transmitters must receive different channels, and any two "very close" transmitters must receive channels at least two apart. One can construct an interference graph for this problem so that the transmitters are represented by the vertices and there is an edge joining two vertices of "very close" transmitters. Two transmitters are defined as "close" if the corresponding vertices are of distance two.

For a given graph G of order n, an $L(2, 1)$-*labelling* is defined as a function $f: V(G) \to \{0, 1, 2, \ldots\}$ such that $|f(u) - f(v)| \geqslant 2$ when $d_G(u, v) = 1$ and $|f(u) - f(v)| \geqslant 1$ when $d_G(u, v) = 2$, where $d_G(u, v)$, the distance between u and v, is the minimum length of a path between u and v. A k-$L(2, 1)$-*labelling* is an $L(2, 1)$-labelling such that no integer is greater than k. The $L(2, 1)$-*labelling number* of G, denoted by $\lambda(G)$, is the smallest number k such that G

☆ Supported in part by National Natural Science Foundation of China (Nos. 10301010 and 60673048).
* Corresponding author. Department of Mathematics, East China Normal University, Shanghai 200062, PR China.
E-mail address: chlu@math.ecnu.edu.cn (C. Lu).

0166-218X/$ - see front matter © 2007 Elsevier B.V. All rights reserved.
doi:10.1016/j.dam.2006.12.002

has a k-$L(2,1)$-labelling. The $L(2,1)$-labelling problem has been extensively studied during the past decade (see [1,7–9,11]).

Another related interesting problem called consecutive 2-distant coloring of a graph was first introduced in [16] under the name "no-hole 2-distant coloring". For a simple graph $G=(V,E)$, a consecutive 2-distant coloring of G is an assignment $f: V \to \{0,1,2,\ldots\}$ such that $|f(u)-f(v)|\geqslant 2$ when $d_G(u,v)=1$ and $\{f(v): v \in V\}$ is a set of consecutive integers. We call $sp(G,f)=\max\{f(u)-f(v): u,v \in V\}$ the span of f. If G admits a consecutive 2-distant coloring, then define $csp(G)=\min sp(G,f)$ with minimum taken over all such colorings f. The reader is referred to [2,3,12,15–19] for recent work concerning consecutive 2-distant colorings.

Motivated by concepts of $L(2,1)$-labelling and consecutive 2-distant coloring of graph, in this paper we will focus on channel assignments under the following constraints: (a) neighboring transmitters use channels that differ by at least 2; (b) transmitters with distance two use channels that differ by at least 1; (c) channels used consist of a set of consecutive integers. The *consecutive* $L(2,1)$-*labelling* is a variation of $L(2,1)$-labelling under the condition that the labelling f is an onto function. The definition of the *consecutive* $L(2,1)$-*labelling number* $\overline{\lambda}(G)$ is the same as that of the $L(2,1)$-labelling number except that the integers used are consecutive. The concept of consecutive $L(2,1)$-labelling of graphs was first introduced in [6] under the name "no-hole $L(2,1)$-colorings". Some results on consecutive $L(2,1)$-labelling of graphs can be found in [5,6].

Obviously, many graphs do not admit a consecutive $L(2,1)$-labelling. For example, any complete graph K_n (with $n\geqslant 2$) does not admit consecutive $L(2,1)$-labelling. From prior works in [8,16,6], the existence of consecutive $L(2,1)$-labelling of graphs can be established by the following theorem, which shows that it is closely related to the consecutive 2-distant coloring and the $L(2,1)$-labelling. This observation is also one of the motivations of studying consecutive $L(2,1)$-labelling.

Theorem 1. *For any graph G of order n, the following four statements are equivalent*:

(1) *G admits a consecutive* $L(2,1)$-*labelling*.
(2) *G admits a consecutive* 2-*distant coloring*.
(3) *The complement graph* G^c *has a Hamilton path*.
(4) $\lambda(G)\leqslant n-1$.

Let G be a graph of order n. If G admits a consecutive $L(2,1)$-labelling, then we easily know that

$$\lambda(G)\leqslant\overline{\lambda}(G)\leqslant n-1. \tag{1}$$

The main purpose of this paper is to investigate the graphs with $\overline{\lambda}(G)=n-1$ and the graphs with $\overline{\lambda}(G)=\lambda(G)$. In Section 2, we study the Hamiltonicity of the complement graph of G, which plays an important role in the proof of the main results in Section 4. In Section 3, a theorem on the structure of graphs with $\lambda(G)\neq\overline{\lambda}(G)$ is raised, which is essential to the proof of the main results in Section 5. In Section 4, we investigate the graphs of order n with $\overline{\lambda}(G)=n-1$, in terms of their sizes, diameters and the number of components. In Section 5, we investigate the graphs with $\overline{\lambda}(G)=\lambda(G)$, in terms of their sizes, diameters and the number of components.

Here, we first introduce some notation and terminology. Let $G=(V(G),E(G))$ be a finite, undirected graph. For $v\in V(G)$, $N_G(v)$ is the set of neighbors of v in G, and the degree of vertex v in graph G, written $d_G(v)$, is the number of neighbors of v in G. The maximum degree is Δ, the minimum degree is δ. When $S\subseteq V(G)$, the induced subgraph $G[S]$ consists of S and all edges whose endpoints are contained in S. A matching in a simple graph G is a set of edges with no shared endpoints. A perfect matching of G is a matching that saturates every vertex of $V(G)$. The complement G^c of a simple graph G is the simple graph with vertex set $V(G)$ defined by $uv\in E(G^c)$ if and only if $uv\notin E(G)$. For $v\in V(G)$ (or $e\in E(G)$), $G-v$ (or $G-e$) denotes the subgraph of G obtained by deleting the vertex v (or the edge e, respectively). If H is a subgraph of G, then $G-H$ is the graph with vertex set $V(G)$ and edge set $E(G)-E(H)$. The disjoint union of graphs G and H, denoted by $G\cup H$, is the graph with vertex set $V(G)\cup V(H)$ and edge set $E(G)\cup E(H)$. The path, cycle, complete graph and star with n vertices are denoted by P_n, C_n, K_n and $K_{1,n-1}$, respectively. Let a and b be non-negative integers. If $b\leqslant a$, the binomial coefficient $\binom{a}{b}$ is defined to be the number of b-element subsets of a set of a elements. If $b>a$, we define $\binom{a}{b}=0$. Some other notations and terminology not introduced here can be found in [20].

2. Hamiltonicity of the complement graph

In this section we study the Hamiltonicity of the complement graph of G, which plays an important role in the proof of the main results.

The following lemma can be found in [4,14].

Lemma 2 (*Dirac [4], Ore [14]*). (i) *If G is a simple graph of order $n \geqslant 3$ and $\delta(G) \geqslant n/2$, then G is Hamiltonian.*

(ii) *If G is a simple graph with $\delta(G) \geqslant (n-1)/2$, then G has a Hamilton path.*

(iii) *Let G be a simple graph of order $n \geqslant 3$. If $d_G(x) + d_G(y) \geqslant n$ for any two non-adjacent vertices x and y, then G is Hamiltonian.*

Since $d_{G^c}(x) = n - 1 - d_G(x)$ for any $x \in V(G)$, we easily get the following consequences by Lemma 2.

Lemma 3. (i) *If G is a simple graph of order $n \geqslant 3$ and $\Delta(G) \leqslant n/2 - 1$, then G^c is Hamiltonian.*

(ii) *If G is a simple graph with $\Delta(G) \leqslant (n-1)/2$, then G^c has a Hamilton path.*

(iii) *Let G be a simple graph of order $n \geqslant 3$. If $d_G(x) + d_G(y) \leqslant n - 2$ for any $xy \in E(G)$, then G^c is Hamiltonian.*

Now we will discuss other sufficient conditions for the Hamiltonicity of the complement graph.

Theorem 4. *If G is a simple graph of order $n \geqslant 3$ and size $m \leqslant n - 3$, then G^c is Hamiltonian. If $m \leqslant n - 2$, then G^c has a Hamilton path.*

Proof. If $m \leqslant n - 3$, then $d_G(x) + d_G(y) \leqslant m - 1 + 2 \leqslant n - 2$ for any $xy \in E(G)$. Hence, G^c is Hamiltonian by Lemma 3. If $m = n - 2$, then $G - e$ has $n - 3$ edges for any edge $e \in E(G)$. Thus, the complement graph $(G - e)^c$ has a Hamilton cycle C, which implies that G^c has a Hamilton path whether $e \in C$ or not. □

Theorem 5. *If G is a simple graph of order $n \geqslant 3$ and $C(G) \geqslant \lceil n/2 \rceil + 1$, where $C(G)$ denotes the number of components of G, then G^c is Hamiltonian; if $C(G) \geqslant \lceil (n+1)/2 \rceil$, then G^c has a Hamilton path.*

Proof. Let H be a component with maximum order n_1. If $C(G) \geqslant \lceil n/2 \rceil + 1$ and $n \geqslant 3$, we have $\Delta(G) \leqslant n_1 - 1 \leqslant n - C(G) \leqslant \lfloor n/2 \rfloor - 1$. Thus G^c is Hamiltonian by Lemma 3. If $C(G) \geqslant \lceil (n+1)/2 \rceil$, we have $\Delta(G) \leqslant n - \lceil (n+1)/2 \rceil = \lfloor (n-1)/2 \rfloor$. Thus G^c has a Hamilton path by Lemma 3. □

Theorem 6. *If G is a simple connected graph of order $n \geqslant 3$ and diameter $d \geqslant \lfloor n/2 \rfloor + 2$, then G^c is Hamiltonian. If $d \geqslant \lceil n/2 \rceil + 1$, then G^c has a Hamilton path.*

Proof. Assume $d \geqslant \lfloor n/2 \rfloor + 2$. Let $d_G(u, v) = d$, where $u, v \in V(G)$. We apply a *breadth–first search* (commonly abbreviated as *BFS*) to G rooted at u. Thus, $V(G)$ can be partitioned into $V_0, V_1, \ldots, V_d$ with $d_G(u, x) = i$ for any vertex $x \in V_i$ for $i = 0, 1, \ldots, d$. Clearly, $V_0 = \{u\}$ and $v \in V_d$, moreover, $N_G(V_i) \subseteq V_{i-1} \cup V_i \cup V_{i+1}$ for $i = 1, 2, \ldots, d-1$, where $N_G(V_i)$ denotes the unions of neighbors of $x \in V_i$. Hence, $\Delta(G) \leqslant n - d + 1 \leqslant n + 1 - \lfloor n/2 \rfloor - 2 = \lceil n/2 \rceil - 1$ since $d \geqslant \lfloor n/2 \rfloor + 2$. By Lemma 3, G^c is Hamiltonian if n is even.

Thus suppose that n is odd, we only need to consider the case $\Delta(G) = (n-1)/2$ by Lemma 3. Notice that $\Delta(G) = (n-1)/2$ if and only if $d = \lfloor n/2 \rfloor + 2 = (n+3)/2$ and there exists a $u - v$ path P of length d with an internal vertex x in P with degree $(n-1)/2$. This implies that x is adjacent to every vertex in $V(G) \backslash V(P)$. Let $V(P) = \{a_i | i = 0, 1, \ldots, (n+3)/2\}$, where $a_i \in V_i$, $V(G) \backslash V(P) = \{b_1, b_2, \ldots, b_{(n-5)/2}\}$. Suppose that $x = a_k$ $(1 \leqslant k \leqslant d - 1)$, then $\{a_{k-1}, a_k, a_{k+1}, b_1, b_2, \ldots, b_{(n-5)/2}\} = V_{k-1} \cup V_k \cup V_{k+1}$. (See Fig. 1).

Case 1: There exists some b_i with $d_G(b_i) \leqslant (n-3)/2$. Let $H = G - b_i$. Obviously, H is connected and the diameter of H is not less than $d = (n+3)/2$. Note that H has $n-1$ vertices and $n-1$ is even. We know H^c is Hamiltonian by above argument. Let C be a Hamilton cycle of H^c. The assumption $d_G(b_i) \leqslant (n-3)/2$ implies that there exist two consecutive vertices $x, y \in V(C)$ such that $xb_i \notin E(G)$ and $yb_i \notin E(G)$. Thus $C - xy + xb_i + b_i y$ is a Hamilton cycle of G^c.

Case 2: $d_G(b_i) = (n-1)/2$ for any $i \in \{1, 2, \ldots, (n-5)/2\}$. In this case, $a_k, b_1, \ldots, b_{(n-5)/2}$ are pairwise adjacent in G and b_i has exactly three neighbors in path P for any $i \in \{1, 2, \ldots, (n-5)/2\}$. Thus any b_i $(1 \leqslant i \leqslant (n-5)/2)$ is not

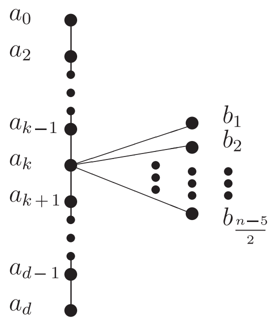

Fig. 1. The graph G with $d_G(a_k) = (n+3)/2$.

in V_d since $b_i \in V_d$ implies that b_i has at most two neighbors in P. Obviously, either $\{a_{k-1}, a_k, b_1, b_2, \ldots, b_{(n-5)/2}\} = V_{k-1} \cup V_k$ or $\{a_k, a_{k+1}, b_1, b_2, \ldots, b_{(n-5)/2}\} = V_{k+1} \cup V_k$ since $b_1, b_2, \ldots, b_{(n-5)/2}$ are pairwise adjacent. Without loss of generality, we assume $\{a_{k-1}, a_k, b_1, b_2, \ldots, b_{(n-5)/2}\} = V_{k-1} \cup V_k$. When $k = d-1$, G^c has a Hamilton cycle: $a_{(n+1)/2}a_{(n-3)/2}a_{(n+3)/2}a_{(n-1)/2}a_0b_1 \ldots a_{(n-7)/2}b_{(n-5)/2}a_{(n-5)/2}a_{(n+1)/2}$ (note that $n \geqslant 7$ in this case). When $1 \leqslant k \leqslant d-2$, G^c has a Hamilton cycle: $a_ka_{k+2}a_{k-1}a_{k+1}a_{i_1}b_1 \ldots a_{i_{(n-5)/2}}b_{(n-5)/2}a_{i_{(n-3)/2}}a_k$, where $i_1, \ldots, i_{(n-3)/2}$ take distinct values from $\{0, 1, \ldots, (n+3)/2\} \backslash \{k-1, k, k+1, k+2\}$.

Now we prove that G^c has a Hamilton path if the diameter of G is not less than $\lceil n/2 \rceil + 1$. By the above, we only need to consider the case that n is even and $d = n/2 + 1$. Similarly, we have $\Delta(G) \leqslant n - d + 1 = n + 1 - (n/2 + 1) = n/2$. By Lemma 3, G^c has a Hamilton path except for the case $\Delta(G) = n/2$. Notice that $\Delta(G) = n/2$ if and only if $d = (n+2)/2$ and there exists a path P of length d in G with an internal vertex x such that $V(G) \backslash V(P) \subset N(x)$. Now randomly select a vertex $y \in V(G) \backslash V(P)$. Let $H = G - y$ and suppose that the diameter of H is d'. Since H is connected, we have $d' \geqslant d = (n+2)/2$. Note that n is even and H has order $n-1$. So $d' \geqslant (n+2)/2 = \lfloor (n-1)/2 \rfloor + 2$, hence H^c has a Hamilton cycle C by the above result. Since $d_G(y) \leqslant \Delta(G) = n/2$ and $|V(C)| = n-1$, there exists at least a vertex z in C with $yz \notin E(G)$. Extending the cycle C to y at z, we obtain a Hamilton path of G^c. $\square$

Remark 7. It is not difficult to check that the conditions proposed in Theorems 4–6 cannot be weakened. Considering that the main purpose of this paper is consecutive $L(2, 1)$-labelling, we omit the illustrations here.

Since graph G admits a consecutive $L(2, 1)$-labelling if and only if G^c has a Hamilton path by Theorem 1, we know:

Corollary 8. *Let G be a graph of order n and size m, and G has $C(G)$ components. Then $\bar{\lambda}(G) \leqslant n-1$ if one of the following holds*:

(i) $m \leqslant n-2$.
(ii) $C(G) \geqslant \lceil (n+1)/2 \rceil$.
(iii) G is a connected graph with diameter $d \geqslant \lceil n/2 \rceil + 1$.

3. A structure theorem on graphs with $\bar{\lambda} \neq \lambda$

Let f be a k-$L(2, 1)$-labelling of G. Let $f^{-1}(i) = \{v \in V(G) | f(v) = i\}$ and let l_i denote the cardinality of $f^{-1}(i)$ for $i = 0, 1, \ldots, k$. We call the integer h, $0 < h < k$, a *hole* of f if and only if $l_h = 0$. Furthermore, if g is a hole of f such that $l_{g-1} = l_{g+1} = 1$ and $uv \in E(G)$ for $u \in f^{-1}(g-1)$ and $v \in f^{-1}(g+1)$, we call g a *gap* of f. We call the integer m a *multiplicity* of f if $l_m \geqslant 2$ and call the integer s a *single* of f if $l_s = 1$. We let $H(f)$, $G(f)$, $M(f)$ and $S(f)$ denote the collections of holes, gaps, multiplicities and singles of f. Let A be the collection of all $\lambda(G)$-$L(2, 1)$-labelling of G. We say that f is a *minimum* $\lambda(G)$-$L(2, 1)$-*labelling* of G if and only if $f \in A$ and f has the minimum number of holes over A. Obviously, $\lambda(G) = \bar{\lambda}(G)$ if and only if there is a minimum $\lambda(G)$-$L(2, 1)$-labelling f of G with $H(f) = \emptyset$.

A path covering of G, denoted by $P(G)$, is a collection of vertex-disjoint paths in G such that each vertex in $V(G)$ is incident to a path in $P(G)$. A minimum path covering of G is a path covering of G with minimum cardinality and the *path covering number* $p(G)$ is the cardinality of a minimum path covering of G. In [8], Georges et al. show that:

Lemma 9 (*George et al.* [8]). *If f is a minimum $\lambda(G)$-$L(2,1)$-labelling of G, then $G(f)$ is empty or $M(f)$ is empty.*

Theorem 10 (*George et al.* [8]). (i) $\lambda(G)\leqslant n-1$ *if and only if* $p(G^c)=1$.
(ii) *Let r be an integer with* $r\geqslant 2$. *Then* $\lambda(G)=n+r-2$ *if and only if* $p(G^c)=r$.

Let f be a $\lambda(G)$-$L(2,1)$-labelling of G. If $h-1$ and h are holes of f, then we can define a $(\lambda(G)-1)$-$L(2,1)$-labelling g as follows:

$$g(v)=\begin{cases} f(v) & \text{if } f(v)\leqslant h-1,\\ f(v)-1 & \text{if } f(v)\geqslant h+1.\end{cases}$$

This contradicts that f is a $\lambda(G)$-$L(2,1)$-labelling of G. Hence, if f is a $\lambda(G)$-$L(2,1)$-labelling of G, the holes of f are not consecutive integers. Suppose that $h_1, h_2, \ldots, h_k$ are holes of a $\lambda(G)$-$L(2,1)$-labelling f of G with $0<h_1<h_2<\cdots<h_k<\lambda(G)$. Hence, $\{0,1,\ldots,\lambda(G)\}$ is separated into $k+1$ segments by k holes. For example, $\{0,1,\ldots,h_1-1\}$ is a segment of f, and $\{h_1+1,h_1+2,\ldots,h_2-1\}$ is another segment of f. Define $f^{-1}(h_1-1)$, $f^{-1}(h_2-1),\ldots,f^{-1}(h_k-1)$ to be *forward walls* of f and $f^{-1}(h_1+1), f^{-1}(h_2+1),\ldots,f^{-1}(h_k+1)$ to be *backward walls* of f. A *wall* of f is either a forward wall or a backward wall of f. It is possible that a wall is both forward wall and backward wall. Hence, for a $\lambda(G)$-$L(2,1)$-labelling f with k holes, there are l walls with $k+1\leqslant l\leqslant 2k$.

Now we give our main result in this section.

Theorem 11. *Let f be a minimum $\lambda(G)$-$L(2,1)$-labelling of G. For any two walls whose labels are not in the same segment, the subgraph induced by them is a perfect matching.*

Proof. Suppose that f has k holes with $0<h_1<h_2<\cdots<h_k<\lambda(G)$ and $k\geqslant 1$. We first show that the subgraph induced by $f^{-1}(h_i-1)\cup f^{-1}(h_i+1)$ is a perfect matching for any $1\leqslant i\leqslant k$. Let $F=f^{-1}(h_i-1)$ and $B=f^{-1}(h_i+1)$. Obviously, the maximum degree of $G[F\cup B]$ is less than 2 by the definition of $L(2,1)$-labelling. If the minimum degree of $G[F\cup B]$ is 0, without loss of generality, we assume that the cardinality of F is not less than the cardinality of B and there exists a vertex $v\in F$ such that its degree in $G[F\cup B]$ is 0. If the cardinality of F is more than one, there exists a $\lambda(G)$-$L(2,1)$-labelling f' of G which is identical to f on $G-v$ and which assigns the integer h_i to v. Thus f' is a $\lambda(G)$-$L(2,1)$-labelling of G with one fewer hole, contradicting the assumption that f is minimum. Hence, both F and B are singles of f. Assume that $F=\{v\}$ and $B=\{u\}$. Since there is no edge between u and v, the mapping f' given by

$$f'(v)=\begin{cases} f(v) & \text{if } f(v)\leqslant h_i-1,\\ f(v)-1 & \text{if } f(v)\geqslant h_i+1\end{cases}$$

is a $(\lambda(G)-1)$-$L(2,1)$-labelling of G, a contradiction.

Let M and N be two walls in different segments of f. We will show that $G[M\cup N]$ is a perfect matching.

Case 1: $M=f^{-1}(h_i-1)$ and $N=f^{-1}(h_j-1)$ with $i<j$. We define a new labelling g as follows:

$$g(v)=\begin{cases} h_i+h_j-f(v) & \text{if } h_i+1\leqslant f(v)\leqslant h_j-1,\\ f(v) & \text{otherwise.}\end{cases}$$

g is also a minimum $\lambda(G)$-$L(2,1)$-labelling of G with k holes and h_i is also a hole of g. (Note that the holes of g may be different from that of f.) We easily find that $M=g^{-1}(h_i-1)$ and $N=g^{-1}(h_i+1)$. As discussed above, we get that $G[M\cup N]$ is a perfect matching.

Case 2: $M=f^{-1}(h_i-1)$ and $N=f^{-1}(h_j+1)$ with $i<j$. Define a new labelling g as follows:

$$g(v)=\begin{cases} h_j-h_i+f(v) & \text{if } f(v)\leqslant h_i-1,\\ f(v)-h_i-1 & \text{if } h_i+1\leqslant f(v)\leqslant h_j-1,\\ f(v) & \text{otherwise.}\end{cases}$$

g is also a minimum $\lambda(G)$-$L(2,1)$-labelling of G with k holes and h_j is also a hole of g. It is clear that $g^{-1}(h_j-1)=M$ and $g^{-1}(h_j+1)=N$. So, we also have that $G[M\cup N]$ is a perfect matching.

Case 3: $M = f^{-1}(h_i + 1)$ and $N = f^{-1}(h_j - 1)$ with $i + 1 < j$. Define a labelling as follows:

$$g(v) = \begin{cases} h_i + h_{j-1} - f(v) & \text{if } h_i + 1 \leqslant f(v) \leqslant h_{j-1} - 1, \\ h_{j-1} + h_j - f(v) & \text{if } h_{j-1} + 1 \leqslant f(v) \leqslant h_j - 1, \\ f(v) & \text{otherwise.} \end{cases}$$

It is easy to check that g is also minimum $\lambda(G)$-$L(2, 1)$-labelling of G with k holes and h_{j-1} is also a hole of g. It is easy to know that $g^{-1}(h_{j-1} - 1) = M$ and $g^{-1}(h_{j-1} + 1) = N$. Hence, $G[M \cup N]$ is a perfect matching.

Case 4: $M = f^{-1}(h_i + 1)$ and $N = f^{-1}(h_j + 1)$ with $i < j$. Define a labelling g as follows:

$$g(v) = \begin{cases} h_i + h_j - f(v) & \text{if } h_i + 1 \leqslant f(v) \leqslant h_j - 1, \\ f(v) & \text{otherwise.} \end{cases}$$

Similarly, we check that g is a minimum $\lambda(G)$-$L(2, 1)$-labelling of G with k holes and h_j is a hole of g. It is easy to know that $g^{-1}(h_j - 1) = M$ and $g^{-1}(h_j + 1) = N$. Hence, $G[M \cup N]$ is a perfect matching. □

In [8], Georges et al. give the following result which is a corollary of Theorem 11.

Corollary 12 (*Georges et al. [8]*). *Let f be a minimum $\lambda(G)$-$L(2, 1)$-labelling of G. If h is a hole of f, then $l_{h-1} = l_{h+1} > 0$.*

Let f be a minimum $\lambda(G)$-$L(2, 1)$-labelling of G with k holes. Then f has $k + 1$ segments. Every segment has at least one wall. Thus, Theorems 11 and 10 lead immediately to the next result, which was first raised by Georges et al. (see [8]).

Corollary 13 (*Georges et al. [8]*). *For $r \geqslant 1$, if $\lambda(G) = n + r - 2$, then G contains a subgraph isomorphic to the complete graph K_r.*

4. The graphs with $\overline{\lambda} = n - 1$

We have learned that $\lambda(G) \leqslant \overline{\lambda}(G) \leqslant n - 1$. In this section we study the graph G with $\overline{\lambda}(G) = n - 1$.

Lemma 14. *If G^c is Hamiltonian and $\overline{\lambda}(G) = n - 1$, then G has at most two components.*

Proof. Let $C = v_0 v_1 \cdots v_{n-1} v_0$ be a Hamilton cycle of G^c. We claim that either $d_G(v_i, v_{i+1}) = 2$ or $d_G(v_i, v_{i+2}) = 1$ for any $0 \leqslant i \leqslant n-1$. (Here, all subscripts are taken modular n.) Otherwise, for some integer i $(0 \leqslant i \leqslant n-1)$, $d_G(v_i, v_{i+1}) \geqslant 3$ and $d_G(v_i, v_{i+2}) \geqslant 2$. We define a new labelling f as follows: $f(v_i) = 0$ and $f(v_{i+j}) = j - 1$ for $j = 1, 2, \ldots, n - 1$. Obviously f is a consecutive $L(2, 1)$-labelling of G while $\mathrm{span}(f) = n - 2$. Hence $\overline{\lambda}(G) \leqslant n - 2$, a contradiction. This claim implies that each component of G contains at least $n/2$ vertices, hence $C(G) \leqslant 2$. □

Theorem 15. *Let G be a simple graph of order n and size m. If $\overline{\lambda}(G) = n - 1$, then $n - 2 \leqslant m \leqslant ((n - 1)(n - 2))/2$. Moreover, for any two integers n, m with $n \geqslant 3$ and $n - 2 \leqslant m \leqslant ((n - 1)(n - 2))/2$, there exists a graph G of order n and size m with $\overline{\lambda}(G) = n - 1$.*

Proof. $\overline{\lambda}(G) = n - 1$ implies G^{C} has a Hamilton path by Theorem 1. Hence G has at most $(n(n - 1)/2) - (n - 1)$ edges, i.e., $m \leqslant ((n - 1)(n - 2))/2$. On the other hand, if $m \leqslant n - 3$, by Theorem 4, G^c has a Hamilton cycle. Since $\overline{\lambda}(G) = n - 1$, by Lemma 14, we have $C(G) \leqslant 2$. This implies that G has at least $n - 2$ edges, a contradiction and hence $m \geqslant n - 2$.

It is easy to know $G' = K_{1,n-2} \cup K_1$ is a graph of order n and size $n - 2$ with $\overline{\lambda} = n - 1$. Let $u \in V(G')$ be the vertex of degree $n - 2$ and $v \in V(G')$ be the isolated vertex. Now we assign an $(n - 1)$-consecutive $L(2, 1)$-labelling f to G' such that $f(u) = 0$, $f(v) = 1$ and remaining vertices are labelled by $2, 3, \ldots, n - 1$, respectively. Hence $\overline{\lambda}(G') = n - 1$. Starting from G' we join edges to those non-adjacent vertex pairs $\{x, y\}$ with $|f(x) - f(y)| \geqslant 2$. Suppose that the new graph H is gained by adding edges to G' as in the above method. Clearly, f is also an $(n - 1)$-consecutive $L(2, 1)$-labelling of H.

Observe that $K_{1,n-2}\cup K_1$ is a spanning subgraph of H and $\overline{\lambda}(K_{1,n-2}\cup K_1)=n-1$. Thus we know that $\overline{\lambda}(H)=n-1$. Since there are exactly $((n-2)(n-3))/2$ non-adjacent vertex pairs $\{x,y\}$ with $|f(x)-f(y)|\geqslant 2$ in $K_{1,n-2}\cup K_1$, it is sufficient to obtain a desired graph of order n and size m with $\overline{\lambda}=n-1$, where $n-1\leqslant m\leqslant (n-1)(n-2)/2$. $\square$

Theorem 16. *Let G be a graph of order n and size m with* $\overline{\lambda}(G)=n-1$. *Then,*

(i) $G\cong K_n-P_n$ *when* $m=((n-1)(n-2))/2$.
(ii) $G\cong P_2\cup P_2$ *or* $K_3\cup K_1\cup K_1$ *or* $K_{1,n-2}\cup K_1$ *when* $m=n-2$.

Proof. (i) Let H denote K_n-P_n. Since H has $(n-1)(n-2)/2$ edges and $H^c=P_n$, then G must be H if $m=(n-1)(n-2)/2$. Otherwise, G^c does not have a Hamilton path and hence G does not admit a consecutive $L(2,1)$-labelling, contradicting $\overline{\lambda}(G)=n-1$.

(ii) If $m=n-2$, then $C(G)\geqslant 2$. Now we first assume $C(G)\geqslant 3$. By Lemma 14, G^c is not Hamiltonian. Since G has size $n-2$, then $d_G(x)+d_G(y)\leqslant n-1$ for any $xy\in E(G)$. Since G^c is not Hamiltonian, by Lemma 3, there exist two adjacent vertices $x_1,x_2\in V(G)$ with $d_G(x_1)+d_G(x_2)=n-1$. Hence, every edge $e\in E(G)$ is incident to either x_1 or x_2. Let G_1 be the component containing x_1 and x_2. Obviously other components of G are all isolated vertices. Hence, $|V(G_1)|\leqslant n-2$ by $C(G)\geqslant 3$. Since $|E(G_1)|=|E(G)|=n-2$, G_1 is not a tree and $|V(G_1)|\geqslant 3$. Let $V(G_1)=\{x_1,x_2,\ldots,x_{n_1}\}$ and $V(G)\backslash V(G_1)=\{y_1,y_2,\ldots,y_{n-n_1}\}$, where $|V(G_1)|=n_1$. If $n_1\leqslant n-3$, then $n-n_1\geqslant 3$. Thus the cycle $x_1y_1x_2y_2x_3\cdots x_{n_1}y_3\cdots y_{n-n_1}x_1$is a Hamilton cycle of G^c. It contradicts the fact that G^c is not Hamiltonian. Thus, $n_1=n-2$. Suppose that $n_1\geqslant 4$. Since G_1 is not tree, either $d_G(x_1)$ or $d_G(x_2)$ is more than three. Hence, there exists a vertex of degree 1 in G, say x_3. Suppose that x_3 is adjacent to x_2, then the cycle $x_3x_1y_1x_2y_2x_4\cdots x_{n-2}x_3$ is a Hamilton cycle of G^c. It is also a contradiction. So $n_1=n-2=3$ and G is $K_3\cup K_1\cup K_1$.

Now we consider the case $C(G)=2$. Since $m=n-2$ and $C(G)=2$, the two components of G are both trees. We denote them by G_1 and G_2, respectively. If neither G_1 nor G_2 is star, then they both admit a consecutive $L(2,1)$-labelling since a tree admits a consecutive $L(2,1)$-labelling if and only it is not a star (see [6]). Thus $\overline{\lambda}(G)=\max\{\overline{\lambda}(G_1),\overline{\lambda}(G_2)\}<n-1$, a contradiction. Assume G_1 is a star but G_2 is not (similarly for G_2 is a star but G_1 is not). Then G_1 admits a $\lambda(G_1)$-$L(2,1)$-labelling f_1 such that 1 is the unique color not used. Since G_2 is not a star, G_2 admits a $\overline{\lambda}(G_2)$-consecutive $L(2,1)$-labelling f_2 and $\overline{\lambda}(G_2)\geqslant 3$. Combining f_1 and f_2, we obtain a consecutive $L(2,1)$-labelling of G and $\overline{\lambda}(G)\leqslant\{\lambda(G_1),\overline{\lambda}(G_2)\}\leqslant n-2$. It is also a contradiction. Thus we assume G_1 and G_2 are both stars. Let $|V(G_1)|=n_1$ and $|V(G_2)|=n_2$. If $n_1\geqslant n_2\geqslant 3$, then G_1 admits an n_1-$L(2,1)$-labelling f_1' such that 1 is the unique color not used by f_1', and G_2 admits an n_2-$L(2,1)$-labelling f_2' such that n_2-1 is the unique color not used by f_2'. Combining f_1' and f_2', we obtain a consecutive $L(2,1)$-labelling of G and $\overline{\lambda}(G)\leqslant n(G_1)\leqslant n-2$, a contradiction. Hence, we know G_1 and G_2 are both stars and one of them is of order at most 2. Without loss of generality, we assume $n_2\leqslant 2$. If $n_2=1$, then G must be $K_{1,n-2}\cup K_1$; if $n_2=2$, then G_2 is P_2. In this case, if $n_1\geqslant 3$, we can check that $\lambda(G)\leqslant n-2$, a contradiction. Thus $n_1\leqslant 2$ and G_1 is P_2 or K_1. $\square$

Theorem 17. *If* $2\leqslant\overline{\lambda}(G)\leqslant n-2$, *then there exists a simple graph* G' *obtained by adjoining some edges to G such that* $\overline{\lambda}(G')=n-1$.

Proof. Let f be a $\overline{\lambda}(G)$-consecutive $L(2,1)$-labelling of G. Assume $\overline{\lambda}(G)\leqslant n-2$, then there exists a multiple color i of f such that $i\neq 0$. (If 0 is the unique multiple color, we can define $f'=\overline{\lambda}(G)-f$ and consider f'.) Suppose that u is a vertex labelled i and v labelled $i-1$, we can join an edge e between u and v, then define g as follows:

$$g(x)=\begin{cases} f(x)+1 & \text{if } f(x)>i,\\ i+1 & \text{if } x=u,\\ f(x) & \text{if } f(x)\leqslant i \quad \text{and } x\neq u.\end{cases}$$

Clearly g is a $(\overline{\lambda}(G)+1)$-consecutive $L(2,1)$-labelling of $G+e$. Thus $\overline{\lambda}(G)\leqslant\overline{\lambda}(G+e)\leqslant\overline{\lambda}(G)+1$. Hence, $\overline{\lambda}$ is increased by at most one by adding an edge as above. Now we explain why $\overline{\lambda}$ can be raised to $n-1$ after finite steps. After k steps, if we get a new graph H_k with $\overline{\lambda}(G)\leqslant\overline{\lambda}(H_k)<n-1$, we can still add an edge as above and get a new graph H_{k+1} with $\overline{\lambda}(G)\leqslant\overline{\lambda}(H_k)\leqslant\overline{\lambda}(H_{k+1})$ since $\overline{\lambda}(H_k)<n-1$. If $\overline{\lambda}(H_{k+1})=n-1$, we are done. If $\overline{\lambda}(H_{k+1})$ is still

less than $n-1$, we can regard H_k as G and add an edge to H_k as above. But there are only finite edges that can be added, and hence we conclude that $\overline{\lambda}$ can be raised to $n-1$ after finite steps. □

Theorem 17 and its proof also imply that:

Theorem 18. *For any two integers m, n with $2 \leqslant m \leqslant n-1$, there exists a graph G with order n and $\overline{\lambda}(G)=m$.*

The following lemma is easy to prove:

Lemma 19. *For $n \geqslant 3$, if G is neither K_n nor $K_n - e$, then $\lambda(G) \leqslant 2n-4$. Moreover, $\lambda(G)=2n-3$ if and only if $G \cong K_n - e$.*

Theorem 20. *If $\overline{\lambda}(G)=n-1$, then $C(G) \leqslant \lceil n/2 \rceil$. Moreover, for every two integers n and m with $n \geqslant 4$ and $1 \leqslant m \leqslant \lceil n/2 \rceil$, there exists a simple graph G of order n, $C(G)=m$ and $\overline{\lambda}(G)=n-1$.*

Proof. If $n=1$, obviously $C(G)=1$. Notice that $\overline{\lambda}(G)=n-1$ and $n \geqslant 2$ imply that $n \geqslant 3$. If $C(G) \geqslant \lceil n/2 \rceil + 1$, by Theorem 5, G^c has a Hamilton cycle. However, we have $C(G) \leqslant 2$ by Lemma 14. It is a contradiction.

For even $n \geqslant 4$, let G be the union of $K_{n/2+1} - e$ and $n/2-1$ isolated vertices. G is of order n and $C(G)=\lceil n/2 \rceil$ with $\overline{\lambda}(G)=n-1$ by Lemma 19. We can assign G an $(n-1)$-consecutive $L(2,1)$-labelling f as follows: two vertices with degree $n/2-1$ are labelled by 0, 1; those vertices with degree $n/2$ are labelled by $3, 5, 7, \ldots, n-1$, respectively, and remaining $n/2-1$ isolated vertices are labelled by $2, 4, 6 \ldots, n-2$, respectively. For odd $n \geqslant 5$, let G be the union of $K_{(n+1)/2}$ and $(n-1)/2$ isolated vertices. G is a graph of order n and $C(G)=(n+1)/2$ with $\overline{\lambda}(G)=n-1$. We can assign G an $(n-1)$-consecutive $L(2,1)$-labelling f as follows: those vertices with degree $(n-1)/2$ are labelled by $0, 2, 4, \ldots, n-1$, respectively; and remaining isolated vertices are labelled by $1, 3, 5, \ldots, n-2$. Then, appropriate number of edges are joined to those non-adjacent pairs of vertices $\{x, y\}$ of G with $|f(x)-f(y)| \geqslant 2$. Note that $C(G)$ is decreased by at most one when we add an edge to G. Hence, we can similarly obtain the desired graphs. □

Theorem 21. *Let G be a graph of order n and $C(G)=\lceil n/2 \rceil$. If $\overline{\lambda}(G)=n-1$, then*

(i) *for $n=4$, $G \cong K_2 \cup K_2$* or *$P_3 \cup K_1$.*
(ii) *For even $n \geqslant 6$, G is the union of $K_{n/2+1} - e$ and $n/2-1$ isolated vertices.*
(iii) *For odd $n \geqslant 3$, G is the union of $K_{(n+1)/2}$ and $(n-1)/2$ isolated vertices.*

Proof. (i) Obviously.

(ii) For even $n \geqslant 6$, $C(G)=n/2 \geqslant 3$. By Lemma 14, G^c is not Hamiltonian since $\overline{\lambda}(G)=n-1$. Let G_1 be the maximum component of G and $|V(G_1)|=n_1$. We claim that G has only one non-trivial component. For otherwise $d_G(x)+d_G(y) \leqslant 2(n_1-1) \leqslant n-2$ for every $xy \in E(G)$. By Theorem 5, G^c is Hamiltonian, a contradiction. We have $n_1 = n/2+1$ since $C(G)=n/2$ and G_1 is the unique non-trivial component.

If $G_1 \cong K_{n/2+1}$, then G does not admit a consecutive $L(2,1)$-labelling since G has only $n/2-1$ isolated vertices. If G_1 is a graph obtained by omitting more than one edge of $K_{n/2+1}$, by Lemma 19, we have $\lambda(G_1) \leqslant 2(n/2+1)-4=n-2$. Note that every $\lambda(G_1)$-$L(2,1)$-labelling f of G_1 has at most $\lfloor \lambda(G_1)/2 \rfloor$ colors not used, we know that f has less than $n/2$ colors not used. However, G has $n/2-1$ isolated vertices, so G admits a consecutive $L(2,1)$-labelling and $\overline{\lambda}(G)=\lambda(G_1) \leqslant n-2$. This contradicts that $\overline{\lambda}(G)=n-1$. Thus, G is the union of $K_{n/2+1} - e$ and $n/2-1$ isolated vertices.

(iii) For odd $n \geqslant 3$, $C(G)=(n+1)/2 \geqslant 2$. If $C(G)=2$, then $n=3$. It is clear that G is the union of K_2 and an isolated vertex. If $C(G) \geqslant 3$, similarly discussed as above, G has only one non-trivial component G_1 and $n/2-1$ isolated vertices, where $|V(G_1)|=n_1=(n+1)/2$. If G_1 is not $K_{(n+1)/2}$, then $\lambda(G_1) < 2((n+1)/2-1)=n-1$. However, as G has $(n-1)/2$ isolated vertices, $\overline{\lambda}(G)=\lambda(G_1) \leqslant n-2$, a contradiction. Hence G is the union of $K_{(n+1)/2}$ and $(n-1)/2$ isolated vertices. □

Let G be a connected graph of order n, diameter d and $\overline{\lambda}(G)=n-1$. Obviously $n \geqslant 4$. If $n=4$, it is easy to know $G \cong P_4$ with diameter 3; if $n=5$, we have $2 \leqslant d \leqslant 4$. Moreover, $d=2$ if and only if $G \cong C_5$; $d=3$ if and only if G is a tree of maximum degree 3; $d=4$ if and only $G \cong P_5$. In general, we have the following theorem.

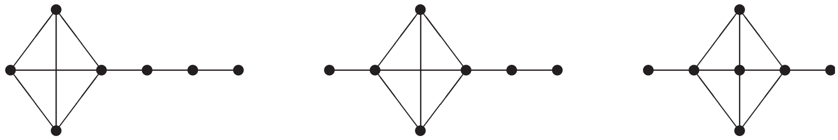

Fig. 2. The graphs of order 7 and diameter 4 with $\overline{\lambda}=6$.

Theorem 22. *Let G be a connected graph of order $n\geqslant 6$ and its diameter be d. If $\overline{\lambda}(G)=n-1$, then $2\leqslant d\leqslant \lfloor n/2\rfloor+1$. Moreover, for every two integers n and d with $n\geqslant 6$ and $2\leqslant d\leqslant \lfloor n/2\rfloor+1$, there exists a simple graph G of order n and diameter d with $\overline{\lambda}(G)=n-1$.*

Proof. Assume to the contrary $n\geqslant 6$ and $d\geqslant \lfloor n/2\rfloor+2$. We first claim that there exists some vertex $x\in V(G)$ such that it is forbidden by at most $n-2$ colors under any $L(2,1)$-labelling of G.

Let $d_G(u,v)=d$ for some $u,v\in V(G)$. Let G be rooted at u, we apply *BFS* to G and partition $V(G)$ into $V_0,V_1,\ldots,V_d$ such that $d_G(u,w)=i$ for any $w\in V_i (i=0,1,\ldots,d)$. Clearly $V_0=\{u\}$ and $v\in V_d$. Let P be a path of length d between u and v, we have $|V(P)|=d+1\geqslant \lfloor n/2\rfloor+3$ and $|V(G)\backslash V(P)|=n-(d+1)\leqslant \lceil n/2\rceil-3$. Hence, $|V_2|\leqslant \lceil n/2\rceil-3-(|V_1|-1)+1=\lfloor (n-1)/2\rfloor-|V_1|$. Thus u is forbidden by at most $3|V_1|+|V_2|\leqslant 2|V_1|+\lfloor (n-1)/2\rfloor$ colors. If $|V_1|\leqslant \lfloor (n-3)/4\rfloor$, then u is forbidden by at most $n-2$ colors. In this case, u is the vertex x in our claim. Assume $|V_1|\geqslant \lfloor (n-3)/4\rfloor+1$. Note that as $n\geqslant 6$ and $d\geqslant \lfloor n/2\rfloor+2\geqslant 5$, the distance of v and v_1 is more than three for any $v_1\in V_1$. Hence, v is forbidden by at most $3(|V(G)\backslash (V(P)\cup V_1)|+1)+1\leqslant 3(\lfloor (n-5)/2\rfloor-\lfloor (n-3)/4\rfloor)+4$ colors. Let $k=3(\lfloor (n-5)/2\rfloor-\lfloor (n-3)/4\rfloor)+4$. For even $n\geqslant 6$, we have $k\leqslant 3((n-6)/2-(n-6)/4)+4=\frac{3}{4}n-\frac{1}{2}\leqslant n-2$; for odd $n\geqslant 9$, we have $k\leqslant 3((n-5)/2-(n-5)/4)+4=\frac{3}{4}n+\frac{1}{4}\leqslant n-2$; for $n=7$, we have $k=4<n-2=5$. Hence, either u or v is the vertex x in our claim.

By Theorem 6, G^{c} has a Hamilton cycle since we assume that $d\geqslant \lfloor n/2\rfloor+2$. Certainly G^{c} has a Hamilton path P with endpoint x. Now label the vertices along path $P-x$ as $0,1,\ldots,n-2$ in turn. At last we can select a color in $\{0,1,\ldots,n-2\}$ to label x since x is forbidden by at most $n-2$ colors. Thus $\overline{\lambda}(G)\leqslant n-2$, a contradiction. Hence, $d\leqslant \lfloor n/2\rfloor+1$.

Now we build the desired graphs of order $n\geqslant 6$ and diameter d with $2\leqslant d\leqslant \lfloor n/2\rfloor+1$.

For even $n\geqslant 6$, let $V(K_{n/2+1})=\{x_1,x_2,\ldots,x_{n/2+1}\}$ and $P_{n/2-1}$ be the path $y_1y_2\cdots y_{n/2-1}$. We first build G_1 by deleting the edge x_1x_2 and adding an edge x_1y_1. Then G_1 is connected with diameter $n/2+1$. Now we prove that $\overline{\lambda}(G_1)=n-1$. It is easy to find that $x_1x_2y_1x_3y_2x_4y_3\cdots x_{n/2}y_{n/2-1}x_{n/2+1}$ is a Hamilton path of G_1^{c}. Then $\overline{\lambda}(G_1)\leqslant n-1$. If $\overline{\lambda}(G_1)=k\leqslant n-2$, then $\lambda(K_{n/2+1}-e)\leqslant k\leqslant n-2$ since $K_{n/2+1}-e$ is an induced subgraph of G_1, where e is the edge x_1x_2. It is a contradiction to Lemma 19. Hence $\overline{\lambda}(G_1)=n-1$. Define a function f as follows: $f(x_1)=0$; $f(x_i)=2i-3$ for $2\leqslant i\leqslant n/2+1$; $f(y_i)=2i$ for $1\leqslant i\leqslant n/2+1$. It is easy to check that f is an $(n-1)$-consecutive $L(2,1)$-labelling of G_1.Adding an edge x_iy_j if $j\neq i+1$ and $j\neq i+2$ or an edge y_iy_j if $|j-i|\geqslant 2$, the new graph gained from G_1 also has $\overline{\lambda}=n-1$ and its diameter may be shortened. By adjoining some suitable edges x_iy_j or y_iy_j to G_1, we can build a graph G of even order n with $\overline{\lambda}(G)=n-1$ and guarantee that the diameter would be shortened by at most one at each step. Hence, we can gain the desired graph G of order $n\geqslant 6$ and diameter d with $\overline{\lambda}(G)=n-1$, where $2\leqslant d\leqslant n/2+1$.

For odd $n\geqslant 7$, let $V(K_{(n+1)/2})=\{x_1,x_2,\ldots,x_{(n+1)/2}\}$ and $P_{(n-1)/2}$ be the path $y_1y_2\cdots y_{(n-1)/2}$. We can build G_2 by adding one edge x_1y_1 to connect $K_{(n+1)/2}$ and $P_{(n-1)/2}$. Then G_2 is connected with diameter $(n+1)/2$. Similarly, we can prove that $\overline{\lambda}(G_2)=n-1$, and adjoining some suitable edges to G_2, we can build the desired graphs for odd order $n\geqslant 7$. The detail is left to readers. □

Remark 23. Different from Theorems 16 and 21, it may be difficult to determine which graphs have diameter $\lfloor n/2\rfloor+1$ and $\overline{\lambda}=n-1$. For example, the graphs in Fig. 2 show that there are at least three graphs with diameter $d=\lfloor n/2\rfloor+1$ and $\overline{\lambda}=n-1$ for $n=7$.

5. The graphs with $\lambda=\overline{\lambda}$

Our main results in this section are the following two theorems.

Theorem 24. *Let G be a graph of order n and size m. If $m \leqslant n-2$, then $\lambda(G)=\overline{\lambda}(G)$ except that G is the disjoint union of several K_2.*

Theorem 25. *Let G be a graph of order n. If $C(G) \geqslant \lceil (n+1)/2 \rceil$, then $\lambda(G)=\overline{\lambda}(G)$.*

In what follows, we prove the above two theorems. Some lemmas are needed.

Lemma 26 (*Griggs and Yeh [9]*). *For any tree T with maximum degree Δ, $\lambda(T)=\Delta+1$ or $\Delta+2$.*

Lemma 27 (*Fishburn and Roberts [6]*). *Let T be a tree of order $n \geqslant 2$. If T is not a star, then $\lambda(T)=\overline{\lambda}(T)$.*

Lemma 28. *Let $G=G_1 \cup T_1 \cup T_2 \cup \cdots \cup T_k$ $(k \geqslant 1)$, where T_i is a tree for $1 \leqslant i \leqslant k$. Let f be a minimum $\lambda(G_1)$-$L(2,1)$-labelling of G_1 with r holes. If $r \leqslant k$ and $\lambda(G_1) > \lambda(T_i)$ for any $1 \leqslant i \leqslant k$, then $\lambda(G)=\overline{\lambda}(G)$.*

Proof. Since $\lambda(G)=\lambda(G_1)$, it suffices to verify that there exists a $\lambda(G)$-$L(2,1)$-labelling f_i for each T_i $(1 \leqslant i \leqslant k)$ such that h_i is used, where $h_1, h_2, \ldots, h_r$ are holes of f.

If T_i is an isolated vertex, then label it with h_i. Suppose that T_i is a tree of maximum degree $\Delta_i \geqslant 1$ and f_i' is a minimum $\lambda(T_i)$-$L(2,1)$-labelling of T_i. By Lemmas 26 and 27, $\lambda(T_i)=\Delta_i+1$ or Δ_i+2, and f_i' has no hole if T_i is not a star.

Case 1: $h_i \geqslant \Delta_i+2$. Define f_i as follows:

$$f_i(v)=\begin{cases} h_i & \text{if } f_i'(v)=\lambda(T_i),\\ f_i'(v) & \text{otherwise.}\end{cases}$$

Case 2: $h_i \leqslant \Delta_i+1$. If T_i is not a star, define $f_i=f_i'$ since f_i' has no hole and h_i has already been used in f_i'. Suppose that T_i is a star. If h_i is used in f_i', define $f_i=f_i'$, we are done. If h_i is a hole of f_i', we label the center of T_i (i.e., the maximum degree vertex of T_i) as h_i, and labels of other vertices are taken from $B=\{0,1,2,\ldots,h_i-2,h_i+2,\ldots,\lambda(G)\}$. Since $\lambda(G) > \lambda(T_i)=\Delta_i+1$, there are at least Δ_i labels in B. It is enough to use them to label the remainder. $\square$

Proof of Theorem 24. Since $m \leqslant n-2$, G is disconnected and it has at least two tree components (i.e., connected component isomorphic to tree). Let $G=G_1 \cup G_2 \cup \cdots \cup G_t \cup T_1 \cup T_2 \cup \cdots \cup T_s$, where G_i is not tree component for $1 \leqslant i \leqslant t$ and T_i is tree component with $1 \leqslant i \leqslant s$ $(s \geqslant 2)$. Let $m_i=|E(G_i)|$ and $n_i=|V(G_i)|$ for $i=1,2,\ldots,t$. Let $m_i'=|E(T_i)|$ and $n_i'=|V(T_i)|$ for $i=1,2,\ldots,s$. It suffices to verify that G admits a minimum $\lambda(G)$-$L(2,1)$-labelling with no holes.

Case 1: $\lambda(G)=\lambda(T_j)$ for some $j \in \{1,2,\ldots,s\}$. Without loss of generality, we assume that $\lambda(G)=\lambda(T_1)$. If T_1 is not a star, then $\lambda(T_1)=\overline{\lambda}(T_1)=\lambda(G)=\overline{\lambda}(G)$. We are done. Suppose that T_1 is a star. We label the center of T_1 as 0 and its neighbors are labelled by $2,3,\ldots,\Delta(T_1)+1$. Hence only 1 is not used. Suppose that $n_1' \geqslant 3$. If T_2 is not a star or T_2 is an isolated vertex, it is clear that we can label 1 at some vertex of T_2. If T_2 is star, we know $\Delta(T_2) \leqslant \Delta(T_1)$ since $\lambda(T_2) \leqslant \lambda(T_1)$. Label T_2 as follows: label the center of T_2 as $\Delta(T_1)+1$ and its neighbors are labelled by $0,1,\ldots,\Delta(T_2)-1$. Then label 1 is used. If $n_1'=2$, then T_1 is K_2 and all G_i disappear for $1 \leqslant i \leqslant t$. Since G is not the disjoint union of several K_2, there exists some T_j $(2 \leqslant j \leqslant s)$ which is an isolated vertex. We label it with 1. We are done.

Case 2: $\lambda(G)=\lambda(G_j)$ for some $j \in \{1,2,\ldots,t\}$. Without loss of generality, we assume that $\lambda(G)=\lambda(G_1)$. If $\lambda(G) \geqslant n_1$, by Theorem 10, $r=p(G_1^{\mathrm{c}})=\lambda(G_1)-n_1+2 \geqslant 2$ and there exists a minimum $\lambda(G_1)$-$L(2,1)$-labelling of G_1 with $r-1$ holes. By Theorem 11 and Corollary 13, G_1 contains a subgraph isomorphic to K_r. Contracting the subgraph isomorphic to K_r to a vertex, we get a new graph G_1' of order n_1-r+1. G_1' is connected since G_1 is connected. Thus G_1' has at least n_1-r edges.

Hence,

$$m_1 \geqslant \binom{r}{2}+n_1-r.$$

Then,

$$m = m_1 + m_2 + \cdots + m_t + m'_1 + m'_2 + \cdots + m'_s \leqslant n - 2,$$

$$\binom{r}{2} + n_1 - r + n_2 + \cdots + n_t + (n'_1 - 1) + \cdots + (n'_s - 1) \leqslant n - 2,$$

$$s \geqslant \binom{r}{2} - r + 2.$$

Thus $s - (r-1) \geqslant \binom{r}{2} - r + 2 - r + 1 = \frac{1}{2}(r-2)(r-3) \geqslant 0$ since $r \geqslant 2$ is an integer. By Lemma 28, we know that $\lambda(G) = \overline{\lambda}(G)$.

If $\lambda(G) = n_1 - 1$, then $p(G_1^c) = 1$ and we label $V(G_1)$ along the Hamilton path in G_1^c. It is an $(n_1 - 1)$-consecutive $L(2, 1)$-labelling of G_1. Hence, $\lambda(G) = \overline{\lambda}(G)$.

Now we assume that $\lambda(G_1) \leqslant n_1 - 2$. Let f be a minimum $\lambda(G_1)$-$L(2, 1)$-labelling of G_1 with k holes. If $k \leqslant 2$, we are done by Lemma 28. Suppose that $k \geqslant 3$ and $h_1, h_2, \ldots, h_k$ are holes of f. We also assume that f has r walls with $k + 1 \leqslant r \leqslant 2k$ and every wall consists of x vertices. Since $\lambda(G_1) \leqslant n_1 - 2$, then $M(f) \neq \emptyset$. By Lemma 9, $G(f) = \emptyset$. Hence $x \geqslant 2$. Let us consider these $k+1$ walls labelled by $h_1 - 1, h_2 - 1, \ldots, h_k - 1, h_k + 1$. Any two of them are not in the same segment of f. Thus, by Theorem 11, there are $\binom{k+1}{2}x$ edges among these $k+1$ walls. By contracting these $k+1$ walls to a vertex, we get a new graph G'_1 of order $n_1 - (k+1)x + 1$. Since G_1 is connected, we know G'_1 is also connected and it has least $n_1 - (k+1)x$ edges.

Thus, we know

$$m_1 \geqslant \binom{k+1}{2} x + (n_1 - (k+1)x).$$

Then,

$$m = m_1 + m_2 + \cdots + m_t + m'_1 + m'_2 + \cdots + m'_s \leqslant n - 2,$$

$$\binom{k+1}{2} x - (k+1)x + n_1 + n_2 + \cdots + n_t + (n'_1 - 1) + \cdots + (n'_s - 1) \leqslant n - 2.$$

Hence,

$$s \geqslant \binom{k+1}{2} x - (k+1)x + 2.$$

Let $g(x) = \binom{k+1}{2}x - (k+1)x + 2$. Since $\binom{k+1}{2} - (k+1) > 0$ for $k \geqslant 3$, $g(x)$ is an increase function. So, $s \geqslant 2\binom{k+1}{2} - 2k$. Since $k \geqslant 3$, then $s - k \geqslant k(k-2) > 0$. By Lemma 28, $\lambda(G) = \overline{\lambda}(G)$. □

Proof of Theorem 25. Let $G = G_1 \cup G_2 \cup \cdots \cup G_c$, where $c = C(G) \geqslant \lceil (n+1)/2 \rceil$. Suppose $\lambda(G) = \lambda(G_1)$ and let f be a minimum $\lambda(G_1)$-$L(2, 1)$-labelling of G_1. If f has no holes, then we are done. Suppose that f has $k \geqslant 1$ holes. Then there are at least $k+1$ walls in f. Hence $n(G_1) \geqslant k + 1$. Suppose there are just s isolated vertices in G. We have $n \geqslant (k+1) + 2(c - s - 1) + s = 2c + k - s - 1$. Since $c \geqslant \lceil (n+1)/2 \rceil$, $s \geqslant 2c + k - n - 1 \geqslant k$. By Lemma 28, we know that $\lambda(G) = \overline{\lambda}(G)$. □

Let G be a connected graph of order n and diameter d. If $d \geqslant \lceil n/2 \rceil + 1$, then $\overline{\lambda}(G) \leqslant n - 1$. (see Corollary 8). If $d \geqslant \lceil n/2 \rceil + 1$, is it true for $\lambda(G) = \overline{\lambda}(G)$? As shown in [6], the graph in Fig. 3 is a graph of order 8 and diameter 5 with $\overline{\lambda} = \lambda + 1 = 6$ (readers can check it easily). Now we show that the graph in Fig. 4 is a graph of order $n = 6k$ and diameter $d = 4k - 1$ with $\overline{\lambda} = \lambda + 1 = 7$ $(k \geqslant 2)$. First we give a 6-$L(2, 1)$-labelling f of G as follows: $f(u_i) = 5$, $f(x_i) = 0$, $f(y_i) = 4$, $f(v_i) = 1$, $f(z_i) = 6$ and $f(w_i) = 2$ for $i = 1, 2, \ldots, k$. Then, $\lambda(G) \leqslant 6$. Since G contains K_4 as its subgraph, we know $\lambda(G) = 6$. Now we show $\overline{\lambda}(G) = 7$. Obviously, $\overline{\lambda}(G) \geqslant \lambda(G) = 6$. If $\overline{\lambda}(G) = 6$, every induced subgraph K_4 of G must be labelled by 0, 2, 4, 6. So, u_i and v_i $(1 \leqslant i \leqslant k)$ must be labelled by 1, 3, 5. We claim that u_i and v_i $(1 \leqslant i \leqslant k)$ cannot be labelled by 3. If not, without loss of generality, we assume that there exists some $i \in \{1, 2, \ldots, k\}$ such that u_i is labelled by 3. Then x_i and w_i must be labelled by 0, 6, and hence y_i and z_i must be labelled by 2, 4. Then, v_i cannot

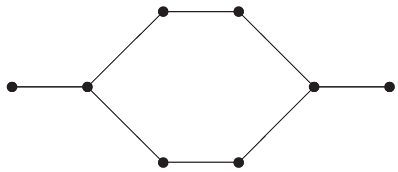

Fig. 3. The graph of order 8 and diameter 5 with $\overline{\lambda} = \lambda + 1 = 6$.

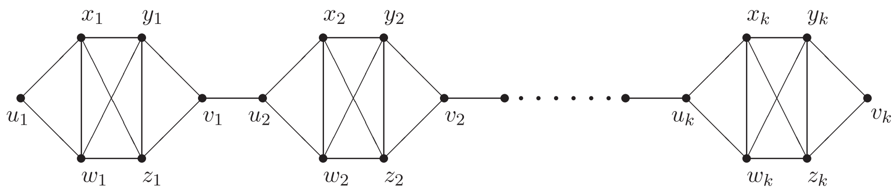

Fig. 4. The graph of order $n = 6k$ $(k \geqslant 2)$ and diameter $d = 4k - 1$ with $\overline{\lambda} = \lambda + 1 = 7$.

be labelled by any integers in $\{0, 1, 2, 3, 4, 5, 6\}$. So, u_j and v_j $(1 \leqslant j \leqslant k)$ must be labelled by 1, 5. Then, 3 cannot be used by any vertex, a contradiction to $\overline{\lambda}(G) = 6$. Hence, $\overline{\lambda}(G) \geqslant 7$. At last we give a 7-consecutive $L(2, 1)$-labelling f of G as follows: for odd i, $f(u_i) = 5$, $f(x_i) = 0$, $f(y_i) = 4$, $f(v_i) = 1$, $f(z_i) = 6$ and $f(w_i) = 2$; for even i, $f(u_i) = 3$, $f(x_i) = 0$, $f(y_i) = 2$, $f(v_i) = 7$, $f(z_i) = 4$ and $f(w_i) = 6$. In general, we may ask the following question.

Question 29. *Is there other connected graph G of order n and diameter* $d \geqslant \lceil n/2 \rceil + 1$ *with* $\lambda(G) \neq \overline{\lambda}(G)$ *except graphs in Figs.* 3 *and* 4?

References

[1] G.J. Chang, D. Kuo, The $L(2, 1)$-labelling problem on graphs, SIAM J. Discrete Math. 9 (1996) 309–316.
[2] G.J. Chang, J.S.T. Juan, D.D.F. Liu, Minimum span of no-hole $(r + 1)$-distant colorings, SIAM J. Discrete Math. 14 (2001) 370–380.
[3] G.J. Chang, J.S.T. Juan, D.D.F. Liu, No-hole 2-distant colorings for unit interval graphs, Ars Combin. 61 (2001) 233–244.
[4] G.A. Dirac, Some theorems on abstract graphs, Proc. London Math. Soc. 2 (1952) 69–81.
[5] P.C. Fishburn, F.S. Roberts, Full color theorems for $L(2, 1)$-colorings, SIAM J. Discrete Math. 20 (2006) 428–443.
[6] P.C. Fishburn, F.S. Roberts, No-hole $L(2, 1)$-colorings, Discrete Appl. Math. 130 (2003) 513–519.
[7] J.P. Georges, D.W. Mauro, M.I. Stein, Labelling products of complete graphs with a condition at distance two, SIAM J. Discrete Math. 14 (2000) 28–35.
[8] J.P. Georges, D.W. Mauro, M. Whittlesey, Relating path covering to vertex labeling with a condition at distance two, Discrete Math. 135 (1994) 103–111.
[9] J.R. Griggs, R.K. Yeh, Labelling graphs with a condition at distance two, SIAM J. Discrete Math. 5 (1992) 586–595.
[10] W.K. Hale, Frequency assignment: theory and applications, Proc. IEEE 68 (1980) 1497–1514.
[11] J. van den Heuvel, R.A. Leese, M.A. Shepherd, Graph labeling and radio channel assignment, J. Graph Theory 29 (1998) 263–283.
[12] D.D.F. Liu, R.K. Yeh, On minimum spans of no-hole T-colorings, in: Proceedings of the 30th Southeastern International Conference on Combinatorics, Graph Theory, and Computing, Boca Raton, FL, 1999, Congr. Numer. 138 (1999) 39–48.
[14] O. Ore, Note on Hamilton circuits, Amer. Math. Monthly 67 (1960) 55.
[15] R.S. Roberts, T-colourings of graphs: recent results and open problems, Discrete Math. 93 (1991) 229–245.
[16] R.S. Roberts, No-hole 2-distant colorings, Math. Comput. Modelling 17 (1993) 139–144.
[17] D. Sakai Troxell, Minimizing the number of holes in 2-distant colorings, Ars Combin. 41 (1995) 77–86.
[18] D. Sakai Troxell, No-hole k-tuple $(r + 1)$-distant colorings, Discrete Appl. Math. 64 (1996) 67–85.
[19] D. Sakai, C. Wang, No-hole $(r + 1)$-distant colorings, Discrete Math. 119 (1993) 175–189.
[20] D.B. West, Introduction to Graph Theory, second ed., Prentice-Hall, Englewood Cliffs, NJ, 2001.